First class functions in constrained second class systems.

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Abstract

Generators of the algebra of first class functions in a system with second class constraints are found. It is shown that first class functions form algebras with respect to the Dirac bracket and pointwise multiplication.The subspace of functions vanishing on constraint surface are ideals of these algebras. The corresponding quotient algebras are isomorphic to the algebras of phase variables in the Dirac bracket formalism. Explicite expressions for generators and brackets of the algebras under consideration are obtained.

1 Introduction

First class functions play an important role in covariant description of constrained second class systems. Hamilton equations of such systems are defined modulo the functions which vanish on constraint surface. In a recent article [1] it was shown that the corresponding quotient Dirac bracket algebra is isomorphic to a quotient of the algebra of first class

functions with respect to the original Poisson bracket. In spite of the algebra of first class functions was introduced long ago [2], its generators and brackets were not described.

In the present article we find explicite expressions for generators of this algebra in a system with second class constraints and compute corresponding Poisson brackets.

We show that first class functions form a Dirac bracket algebra and compute the corresponding bracket. The first class functions which vanish on constraint surface form an ideal of this algebra. This enables us to construct a new quotient algebra with respect to the Dirac bracket. The new algebra is isomorphic to the original quotient Dirac bracket algebra. Isomorphic image of constrained Hamilton equations can be treated as a partial fixing of gauge invariance.

The algebra of functions on phase space modulo the functions which vanish on constraint surface is also an algebra (with respect to the pointwise multiplication).This enables us to obtain a new form of constrained Hamilton equations.

We observe that first class functions form an algebra with respect to the pointwise multiplication.The same is true for these functions modulo the functions which vanish on constraint surface.We show that this quotient algebra is isomorphic to the algebra of phase variables of the original constrained Hamilton equations.

The Letter is organized as follows. In Section 2 we review some properties of constrained Hamilton equations and describe the algebraic connection between these equations and Hamilton equations on quotient spaces. In Section 3 we find an explicite realization of first class functions. The space of first class functions and a quotient are studied as Poisson bracket algebras.In Sections 4,5 these spaces are studied respectively as Dirac bracket algebras and algebras with respect to the pointwise multiplication.

2 Constrained Hamilton equations

Let M be a phase space with the phase variables η_n, $n = 1...2N$, and the Poisson bracket $[\eta_m, \eta_n] = \omega_{mn}(\eta)$. Let $H(\eta)$ be the original hamiltonian and $\varphi_j(\eta), j = 1...2J$, the second class constraints $det[\varphi_j, \varphi_k]|_{\varphi=0} \neq 0$.

The dynamic of the system under consideration is described by the Hamilton equations (see, e.g.,[3])

$$\frac{d}{dt}\eta_n = [\eta_n, H_T], \qquad \varphi_j = 0. \tag{1}$$

Here $H_T = H + \lambda_j \varphi_j$ and functions $\lambda_j = \lambda_j(\eta)$ are defined by the equation

$$[H_T, \varphi_j]|_{\varphi=0} = 0. \tag{2}$$

Using (2) one can write equations (1) as

$$\frac{d}{dt}\eta_n = [\eta_n, H_T]_D, \qquad \varphi_j = 0. \tag{3}$$

Here the Dirac bracket was introduced

$$[g,f]_D = [g,f] - [g,\varphi_j]c_{jk}[\varphi_k,f], \qquad c_{jk}[\varphi_k,\varphi_l] = \delta_{jl}. \tag{4}$$

Let A be the space of functions on M and $\Phi \subset A$ be the subspace of the functions which vanish on constraint surface.It is known that Φ is an ideal and A/Φ is an algebra with respect to the Dirac bracket.

Let $F : A \to A/\Phi$ be the canonical homomorphism

$$F(g) = \{g\}$$

where $\{g\} \in A/\Phi$ is the coset represented by g. The homomorphic image of equations (3) in A/Φ is

$$\frac{d}{dt}\{\eta_n\} = \{[\eta_n, H_T]_D\}. \tag{5}$$

Equations (5) were introduced by Dirac.In the original notations [2] they are written

$$\frac{d}{dt}\eta_n \approx [\eta_n, H_T]_D,$$

where $f \approx g$ means that $f - g \in \Phi$.

Equations (5) can be written as [1]

$$\frac{d}{dt}\{\eta_n\} = [\{\eta_n\},\{H\})]_D. \tag{6}$$

To rewrite these equations in another form we need the proposition:

PROPOSITION 2.1. *Φ is and ideal of A and A/Φ is an algebra with respect to the pointwise multiplication.*

Proof. The proof is straightforward. □

COROLLARY. In A/Φ

$$f(\{\eta\}) = \{f(\eta)\}.$$

From this it follows that Hamilton equations (6) can be written as

$$\frac{d}{dt}\{\eta_n\} = [\{\eta_n\}, H(\{\eta\})]_D \tag{7}$$

and

$$\varphi_j(\{\eta\}) = \{0\}. \tag{8}$$

Equations (8) tell us that cosets $\{\eta_n\}, n = 1, \ldots, 2N$, represent the phase variables on constraint surface.

Dirac bracket algebra A/Φ is isomorphic to Poisson bracket algebra Ω/Υ [1].Here Ω is the algebra of first class functions and $\Upsilon = \Omega \cap \Phi$. The image of Hamilton equations (7) in Ω/Υ is

$$\frac{d}{dt}\{\tilde{\eta}_n\}^\bullet = [\{\tilde{\eta}_n\}^\bullet, \{H_T\}^\bullet]. \tag{9}$$

Here

$$\tilde{\eta}_n = \eta_n - [\eta_n, \varphi_j] c_{jk} \varphi_k \tag{10}$$

and $\{g\}^\bullet \in \Omega/\Upsilon$ denotes the class represented by $g \in \Omega$.

3 Poisson bracket algebras of first class functions

To describe elements of Ω explicitly let us consider the equations

$$[\tilde{g}, \varphi_i] \in \Phi \tag{11}$$

with the initial condition

$$\tilde{g} \in \{g\}. \tag{12}$$

It is easy to see that the function $u_{ij}(\eta)\varphi_i\varphi_j$ satisfies (11). Hence a solution to these equations can be represented in the form

$$\tilde{g} = g + l_i \varphi_i + u_{ij} \varphi_i \varphi_j \tag{13}$$

for some $l_i, u_{ij} \in A$

Substituting (13) into (11) we find $l_i = -[g, \varphi_i]c_{ij}$. Hence

$$\tilde{g} = L(g) + u \tag{14}$$

Here $L(g) = g - [g, \varphi_i]c_{ij}\varphi_j, u = u_{ij}\varphi_i\varphi_j$ and $u_{ij}(\eta)$ are arbitrary functions.

In particular for $\tilde{v} \in \{0\}$ we have

$$\tilde{v} = v_{ij}\varphi_i\varphi_j. \tag{15}$$

Here $v_{ij}(\eta)$ are arbitrary functions.

Taking into account equations (14,15), we have the proposition:

PROPOSITION 3.1. *Algebras* Ω, Υ *consist of all possible expressions* (14) *and* (15) *respectively, where* $f, u_{ij}, v_{ij} \in A$.

Let

$$\tilde{g}_a = L(g_a) + u_a, \qquad u_a \in \Upsilon, \tag{16}$$

$a = 1, 2$, be first class functions and

$$l_{aj} = -[g_a, \varphi_i]c_{ij}.$$

PROPOSITION 3.2. *The Poisson bracket for first class functions* (16) *is given by*

$$[\tilde{g}_1, \tilde{g}_2] = L([g_1, g_2]_D) + \tilde{u}_{12},$$

$$\tilde{u}_{12} = [l_{1i}, l_{2j}]\varphi_i\varphi_j + [L(g_1), u_2] + [u_1, L(g_2)] + [u_1, u_2] \in \Upsilon.$$

Proof. Straightforward calculation. □

This proposition my be partly checked as follows. It is easy to see that $[\tilde{g}_1, \tilde{g}_2] \in \{[g_1, g_2]_D\}$. On the other hand $[\tilde{g}_1, \tilde{g}_2]$ is a first class function and hence it may be written in the form (14)

$$[\tilde{g}_1, \tilde{g}_2] = L([g_1, g_2]_D) + v, \qquad v \in \Upsilon.$$

COROLLARY *For* $\{\tilde{g}_1\}^\bullet, \{\tilde{g}_2\}^\bullet \in \Omega/\Upsilon$ *we have*

$$[\{\tilde{g}_1\}^\bullet, \{\tilde{g}_2\}^\bullet] = \{L([g_1, g_2]_D)\}^\bullet. \tag{17}$$

Let us compare the functions $\tilde{g}(\eta)$ (14), $g(\tilde{\eta})$ and $\tilde{g}(\tilde{\eta})$ where $\tilde{\eta}_n \in \Omega$ is given by (10). One can check that $g(\tilde{\eta})$ and $\tilde{g}(\tilde{\eta})$ as well as $\tilde{g}(\eta)$ are first class functions.The initial conditions for these functions read $\tilde{g}(\eta), g(\tilde{\eta}), \tilde{g}(\tilde{\eta}) \in \{g(\eta)\}$ and hence

$$\tilde{g}(\eta) = g(\tilde{\eta}) + u = \tilde{g}(\tilde{\eta}) + v$$

for some $u, v \in \Upsilon$.

From this it follows

$$\{\tilde{g}(\eta)\}^\bullet = \{g(\tilde{\eta})\}^\bullet = \{\tilde{g}(\tilde{\eta})\}^\bullet. \tag{18}$$

Due to equation (2) Hamiltonian H_T is a first class function. Using (18) we have

$$\{H_T(\eta\}^\bullet = \{H_T(\tilde{\eta})\}^\bullet. \tag{19}$$

4 Dirac bracket algebras of first class functions

PROPOSITION 4.1.

(i)Ω *is an algebra with respect to the Dirac bracket.*

(ii)Υ *is and ideal of* Ω *and* Ω/Υ *is an algebra with respect to the Dirac bracket.*

Proof. For $g, f \in \Omega$ the first term in the r.h.s. of equation (4) is a first class function. The second one is quadratic in constraints and hence is also a first class function.This proves the first statement.

For $g \in \Omega$ and $f \in \Upsilon$ we have $[g, f]_D \in \Upsilon$.Hence Υ is an ideal of Ω and Ω/Υ is an algebra with respect to the Dirac bracket. □

PROPOSITION 4.2. *The Dirac bracket for first class functions* (16) *is given by*

$$[\tilde{g}_1, \tilde{g}_2]_D = L([g_1, g_2]_D) + \tilde{v}_{12},$$

$$\tilde{v}_{12} = [l_{1i}, l_{2j}]_D \varphi_i \varphi_j + [L(g_1), u_2]_D + [u_1, L(g_2)]_D + [u_1, u_2]_D \in \Upsilon.$$

Proof. Straightforward calculation. $\square$

COROLLARY. *The Dirac bracket for* $\{\tilde{g}_1\}^\bullet, \{\tilde{g}_2\}^\bullet \in \Omega/\Upsilon$ *is given by*

$$[\{\tilde{g}_1\}^\bullet, \{\tilde{g}_2\}^\bullet]_D = \{L([g_1, g_2]_D)\}^\bullet. \tag{20}$$

PROPOSITION 4.3. *Dirac bracket algebra* Ω/Υ *is isomorphic to Poisson bracket algebra* Ω/Υ.

Proof. From equations (17) and (20) we have

$$[\{\tilde{g}_1\}^\bullet, \{\tilde{g}_2\}^\bullet] = [\{\tilde{g}_1\}^\bullet \{\tilde{g}_2\}^\bullet]_D. \tag{21}$$

This proves the statement. $\square$

By using the result of [1] one obtains the following corollary:

COROLLARY.*Dirac bracket algebras* Ω/Υ *and* A/Φ *are isomorphic.*

The realization of Hamilton equations (6) in Dirac bracket algebra Ω/Υ is

$$\frac{d}{dt}\{\tilde{\eta}_n\}^\bullet = [\{\tilde{\eta}_n\}^\bullet, \{H_T\}^\bullet)]_D. \tag{22}$$

Due to (21) equations (22) and (9) are identical.

5 Algebras of first class functions

In this section we study the spaces under consideration as algebras with respect to the pointwise multiplication.

PROPOSITION 5.1.

(i) Ω *is an algebra with respect to the pointwise multiplication.*

(ii)Υ *is and ideal of* Ω *and* Ω/Υ *is an algebra with respect to the pointwise multiplication.*

Proof. The proof is straightforward. $\square$

COROLLARY 1 (Leibniz rule). *For* $\{f\}^\bullet, \{g\}^\bullet, \{h\}^\bullet \in \Omega/\Upsilon$ *one has*

$$[\{f\}^\bullet, \{g\}^\bullet\{h\}^\bullet] = [\{f\}^\bullet, \{g\}^\bullet]\{h\}^\bullet + \{g\}^\bullet[\{f\}^\bullet, \{h\}^\bullet].$$

COROLLARY 2. *In* Ω/Υ

$$g(\{\tilde{\eta}\}^\bullet) = \{g(\tilde{\eta})\}^\bullet.$$

Using COROLLARY 2 and equation (19) one can rewrite Hamilton equations (9) in the form

$$\frac{d}{dt}\{\tilde{\eta}_n\}^\bullet = [\{\tilde{\eta}_n\}^\bullet, H_T(\{\tilde{\eta}\}^\bullet)].$$

PROPOSITION 5.2. *For* $\tilde{g}_1, \tilde{g}_2$ (16) *we have*

$$\tilde{g}_1\tilde{g}_2 = L(g_1g_2) + w_{12},$$

$$w_{12} = l_{1i}l_{2j}\varphi_i\varphi_j + \tilde{g}_1u_2 + u_1\tilde{g}_2 \in \Upsilon.$$

Proof. Straightforward calculation. □

COROLLARY. *For* $\{\tilde{g}_1\}^\bullet, \{\tilde{g}_2\}^\bullet \in \Omega/\Upsilon$ *we have*

$$\{\tilde{g}_1\}^\bullet\{\tilde{g}_2\}^\bullet = \{L(g_1g_2)\}^\bullet.$$

PROPOSITION 5.3. Ω/Υ *and* A/Φ *are isomorphic with respect to the pointwise multiplication.*

Proof. Define the linear function $T : \Omega/\Upsilon \to A/\Phi$

$$T(\{g\}^\bullet) = \{g\}. \qquad (23)$$

This function has the inverse and hence determines the one-to-one correspondence between elements of Ω/Υ and A/Φ [1].

Using the definitions of $\Omega/\Upsilon, A/\Phi$ and T we have

$$T(\{g\}^\bullet\{f\}^\bullet) = T(\{gf\}^\bullet) = \{gf\} = \{g\}\{f\} = T(\{g\}^\bullet)T(\{f\}^\bullet).$$

and hence T is a homomorphism. This proves the statement. $\square$

In equation (23) $\{g\}^\bullet \subset \{g\}$. Hence transition from A/Φ to Ω/Υ can be treated as a partial fixing of gauge invariance.

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References

[1] Bratchikov,A.V.: *Lett.Math.Phys.* **61** (2002) 107.

[2] Dirac,P.A.M. : *Lectures on Quantum Mechanics*, Yeshiva University, New York, 1964.

[3] Gitman, D.M. and Tyutin, I.V.:*Canonical Quantization of Fields with Constraints*, Nauka, Moscow,1986 (in Russian), Springer Series in Nuclear Particle Phys.,Springer-Verlag, Berlin,1990.